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Energy Balance of Uniformly Accelerated Charge*†

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If the electromagnetic field of a uniformly accelerated charge is computed by means of Liénard-Wiechert potentials, it is found that the result does not correspond to a single source. Besides the original uniformly accelerated charge there is also a second charge, which is spread on a plane recessing in the opposite direction with the velocity of light. The work performed by this second charge against the electromagnetic field is equal to the constant rate of radiation of the physical system. This resolves the old paradox of the energy balance of a uniformly accelerated charge, which radiates at a constant rate, although it undergoes no radiation reaction.

I. INTRODUCTION

One of the most puzzling problems of classical physics has been the energy balance of a uniformly accelerated charge (*1–14*). According to the Dirac equation of motion (*15*), a uniformly accelerated charge undergoes no radiation reaction, i.e., it does not perform work against its own electromagnetic field. On the other hand, a standard formula to be found in any textbook (*16*) implies that a charge will radiate whenever its acceleration does not vanish. The question then is: where does the radiated energy comes from?

If the duration of the acceleration is finite, then it can easily be shown that the *total* radiated energy is equal to the *total* work performed against the radiation reaction (*16*) so that a noninstantaneous (nonlocal in time) energy balance can still be written (*11*). However, one may also ask what happens if the duration of the acceleration is formally extended to infinity (into the distant past) even though this requires unrealistic boundary conditions.

This problem is solved here in two steps. First, we compute the total energy of the electromagnetic field

$$W = (1/8\pi)\int(E^2 + H^2)\,dV, \tag{1}$$

and we find that its rate of increase in the instantaneous rest frame is finite and

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invariant, in full agreement with the standard formula for the rate of radiation (*16*):

$$dW/dt = 2e^2/3\alpha^2, \tag{2}$$

where e is the charge and α^{-1} is the acceleration. (We use natural units $c = 4\pi\epsilon_0 = 1$.)

The above calculation makes no reference to the Poynting theorem, and thereby avoids all the recent controversy about how the Poynting vector should be used (*9–12*). However, it becomes conspicuous that the energy increase does not come from the uniformly accelerated charge, but originates at the (finite) discontinuity of the electromagnetic field on a plane situated at a distance α from the charge (as measured in the instantaneous rest frame of the latter).

In the rest frame of the observer, this plane is seen recessing from the charge with the velocity of light, and it is found that it is endowed with a finite charge density. (Its total charge is $-e$.) *It is the work performed by this second charge against the electromagnetic field which is equal to the constant rate of radiation of the physical system.*

Finally, it is shown that this second charge can be eliminated only at the expense of introducing an *infinite* discontinuity of the electromagnetic field, instead of a finite one. The latter then acts as an infinite energy reservoir and the energy balance is again—though only formally—verified.

II. TOTAL ENERGY

The detailed calculations are lengthy but straightforward. We assume that the trajectory of the charge is given by

$$z = (\alpha^2 + t^2)^{1/2}, \tag{3}$$

and its Liénard-Wiechert potentials then lead to the following expression for the electromagnetic field (*9*)

$$E_z = e[-4\alpha^2(\alpha^2 + t^2 + r^2 - z^2)/s^3]\theta(z + t), \tag{4a}$$

$$E_r = e[8\alpha^2 rz/s^3]\theta(z + t), \tag{4b}$$

$$H_\phi = e[8\alpha^2 rt/s^3]\theta(z + t), \tag{4c}$$

$$E_\phi = H_r = H_z = 0, \tag{4d}$$

where we have used cylindrical coordinates $r\phi z$ and where

$$s = [(\alpha^2 + t^2 - r^2 - z^2)^2 + 4\alpha^2 r^2]^{1/2}. \tag{5}$$

As usual,

$$\theta(x)\begin{cases} = 0 & x < 0, \\ = \tfrac{1}{2} & x = 0, \\ = 1 & x > 0. \end{cases} \tag{6}$$

It then turns out that the total energy, as given by Eq. (1), diverges near the charge. We thus introduce, as in the electrostatic case (*17*), a cutoff radius a, as measured in the instantaneous rest frame of the charge, within which the field is assumed to vanish. It is readily shown that the charge distribution on the surface of the cutoff sphere is not spherically symmetric, but the deviation from symmetry is of the order of a/α, which can be made as small as we wish.

The domain of integration for Eq. (1) is now shown by Fig. 1. The authors have been unable to perform the explicit integration for arbitrary t, but in the special case $t = 0$, the integration becomes much simpler and the result is

$$W = e^2\left[\frac{1}{2a} - \frac{a}{16\alpha^2 - 4a^2} + \frac{3}{16\alpha}\ln\frac{2\alpha - a}{2\alpha + a}\right]. \tag{7}$$

The first term is the familiar electrostatic energy of a charge at rest (*17*). The other terms are negligible in the limit $a/\alpha \to 0$.

The next step is the calculation of dW/dt, which can again be computed explicitly only for $t = 0$. The result,

$$dW/dt = 2e^2/3\alpha^2, \tag{8}$$

agrees with the standard formula for the rate of radiation (*16*).

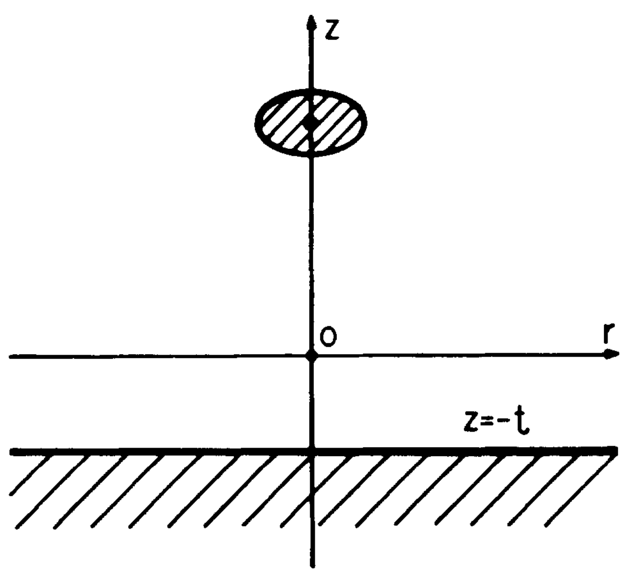

FIG. 1. The domain of integration for Eq. (1) is the nonshaded part of the diagram. The cutoff sphere is centered at $z = (\alpha^2 + t^2)^{1/2}$. Note that it must be flattened in the ratio $\alpha/(\alpha^2 + t^2)^{1/2}$, because of the Lorentz contraction.

III. THE PLANE OF DISCONTINUITY

A remarkable feature of the last formula is that it is cutoff independent. This could have been foreseen without explicit calculations, simply by noting that the integrand in (1) is an even function of t, and therefore the only contribution to dW/dt, for $t = 0$, is due to the motion of the limits of integration. Moreover, for $t = 0$, the charge is at rest, so that only the motion of the plane $z = -t$ contributes to dW/dt. We thus see that the radiated energy comes from the plane $z = -t$, and not from the uniformly accelerated charge.

Let us focus our attention on that plane. First, we note that E_z is discontinuous, so that there is a surface charge density

$$\sigma = \Delta E_z/4\pi = -e\alpha^2/\pi(\alpha^2 + r^2)^2, \tag{9}$$

and a corresponding current density $j_z = -\sigma\delta(z + t)$. A simple calculation shows that the total charge on the plane is

$$\iint \sigma r dr d\phi = -e, \tag{10}$$

and the total work performed by that current is

$$-\iiint j_z E_z r dr d\phi dz = 2e^2/3\alpha^2, \tag{11}$$

in agreement with (8). This resolves the energy balance puzzle.

One may say, however, that this answer is not satisfactory because the physical situation here is not the one that we originally intended to describe. It is naturally tempting to try to eliminate the second charge by placing on the plane $z = -t$ an opposite charge density.

In order to calculate its field, we recall the well known fact (see, e.g., ref. *16*, p. 291) that when a charge moves at a very high velocity $u \approx 1$, its field is increased in a direction at right angles to the direction of motion in the ratio of $(1 - u^2)^{-1/2}$, while in the direction of motion the field is decreased in the ratio $(1 - u^2)$. At very high velocities, the field thus resembles more and more the field of a plane wave. In our case, we have $u = 1$, and the field must have the form of an infinite shock wave of the type $\delta(z + t)$.

Symmetry arguments then lead us to

$$E_r = -H_\phi = f(r)\delta(z + t). \tag{12}$$

(The magnitude of H_ϕ must be equal to that of E_r, since we have a plane wave, and the relative sign must be such that the energy flows in the $-z$ direction.) The explicit value of $f(r)$ is easily obtained by requiring the Gauss theorem to hold for a cylindrical slice enclosing a circle of radius r in the plane $z = -t$. Namely, we have

$$\iint E_r d\phi dz = 4\pi \iint (-\sigma) r dr d\phi, \tag{13}$$

where σ is given by (9), whence

$$f(r) = 2er/(r^2 + \alpha^2). \tag{14}$$

If we now add (12) to (4b) and (4c), the result is identical with the solution which was obtained by Bondi and Gold (*8*), following a rather complicated limiting process. This solution, however, does not seem physically meaningful because of the δ functions. For instance, no proper energy balance can be written for the system, because the total energy stored in the plane $z = -t$ is infinite.

On the other hand, the well behaved solution (4) does not correspond to a single uniformly accelerated charge, but to a pair of charges. We are therefore led to the conclusion that the Maxwell equations are incompatible with the existence of a single charge uniformly accelerated at all times.

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